

# On fundamental groups of tensor product $\text{II}_1$ factors

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## Abstract

Let  $M$  be a  $\text{II}_1$  factor and let  $\mathcal{F}(M)$  denote the fundamental group of  $M$ . In this article, we study the following property of  $M$ : for arbitrary  $\text{II}_1$  factor  $B$ , we have  $\mathcal{F}(M \bar{\otimes} B) = \mathcal{F}(M)\mathcal{F}(B)$ . We prove that for any subgroup  $G \leq \mathbb{R}_+^*$  which is realized as a fundamental group of a  $\text{II}_1$  factor, there exists a  $\text{II}_1$  factor  $M$  which satisfies this property and whose fundamental group is  $G$ . Using this, we deduce that if  $G, H \leq \mathbb{R}_+^*$  are realized as fundamental groups of  $\text{II}_1$  factors (with separable predual), then so are groups  $G \cdot H$  and  $G \cap H$ .

## 1 Introduction and main theorems

In their pioneering work, Murray and von Neumann introduced the fundamental group as an invariant of  $\text{II}_1$  factors [MV43]. For a  $\text{II}_1$  factor  $M$  with trace  $\tau$ , the *fundamental group* is defined as

$$\mathcal{F}(M) := \left\{ \frac{\tau(p)}{\tau(q)} \in \mathbb{R}_+^* \mid p, q \text{ are projections in } M \text{ with } pMp \simeq qMq \right\}.$$

Murray and von Neumann proved the AFD (or *amenable*)  $\text{II}_1$  factor has the full fundamental group  $\mathbb{R}_+^*$ , and then asked the general behavior of this invariant. Indeed, the fundamental group is the most well known invariant for  $\text{II}_1$  factors, and to determine which subgroup of  $\mathbb{R}_+^*$  appears as a fundamental group is a long-standing open problem in the von Neumann algebra theory.

Computation of fundamental groups, however, is a hard problem since  $\text{II}_1$  factors  $pMp$  and  $qMq$  share a lot of properties in common and very difficult to distinguish. Thus very few computation results were known until recently. Connes proved that  $L\Gamma$ , where  $\Gamma$  is an ICC property (T) group, has a countable fundamental group [Co80], which is the first example of a  $\text{II}_1$  factor with fundamental group not equal to  $\mathbb{R}_+^*$ . Voiculescu and Rădulescu proved  $\mathcal{F}(L\mathbb{F}_\infty)$  has the full fundamental group  $\mathbb{R}_+^*$  [Vo89, Ra91].

In 2001, Popa introduced a new framework to study this problem [Po01]. He developed a way of identifying Cartan subalgebras and then reduced the computation problem for  $\text{II}_1$  factors to the one for corresponding orbit equivalence relations. Thus combined with Gaboriau's work on orbit equivalence relations [Ga99, Ga01], Popa obtained the first example of a  $\text{II}_1$  factor which has the trivial fundamental group.

Much progress in the von Neumann algebra theory has been made by this new technology, and the study in this new framework is now called the *deformation/rigidity theory*. Thus, a lot of computations of fundamental groups have been done in the last decade.

We say that a subgroup  $G \leq \mathbb{R}_+^*$  is in the *class*  $\mathcal{S}_{\text{factor}}$  if there is a  $\text{II}_1$  factor  $M$  with separable predual such that  $\mathcal{F}(M) = G$ . Popa proved that any countable subgroup

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of  $\mathbb{R}_+^*$  is contained in  $\mathcal{S}_{\text{factor}}$  [Po03]. Popa and Vaes proved that  $\mathcal{S}_{\text{factor}}$  contains many uncountable subgroups in  $\mathbb{R}_+^*$  [PV08a]. See [Po04, IPP05, Po06a, Ho07, PV08b, De10] for other calculations of fundamental groups. We note that, while this new theory provides a lot of examples, very few general properties for the class  $\mathcal{S}_{\text{factor}}$  are known so far.

The aim of this article is to study fundamental groups of tensor product  $\text{II}_1$  factors. For this, recall that for a  $\text{II}_1$  factor  $M$  and  $t > 0$ , the *amplification*  $M^t$  is defined (up to  $*$ -isomorphism) as  $pMp \overline{\otimes} \mathbb{M}_n$  for any  $n \in \mathbb{N}$  with  $t \leq n$  and any projection  $p \in M$  with trace  $t/n$ . It is then easy to verify that

- $\mathcal{F}(M) = \{t \in \mathbb{R}_+^* \mid M \simeq M^t\};$
- $(M_1 \overline{\otimes} M_2)^{st} \simeq M_1^s \overline{\otimes} M_2^t$  for  $\text{II}_1$  factors  $M_i$  and  $s, t > 0$ .

They particularly imply

$$\mathcal{F}(M_1 \overline{\otimes} M_2) \supset \mathcal{F}(M_1)\mathcal{F}(M_2)$$

for any  $\text{II}_1$  factors  $M_1$  and  $M_2$ . If this inclusion is an equation, then one has a quite useful formula since the computation problem for tensor product  $\text{II}_1$  factors completely reduce to the one for each tensor component. So it is a natural question to ask when such an equation holds true.

In the deformation/rigidity theory, Ozawa and Popa provided the first class of  $\text{II}_1$  factors that satisfy this equality. They proved that if each  $M_i$  is a free group factor, then the tensor product satisfies a *unique prime factorization* result and particularly the equation above holds true [OP03]. See [Pe06, Sa09, CSU11, SW11, Is14, CKP14, HI15, Ho15] for other classes of factors which satisfy the unique prime factorization result.

In this article, we further develop Ozawa–Popa’s strategy, and study the following property of a  $\text{II}_1$  factor  $M$ :

$$\mathcal{F}(M \overline{\otimes} B) = \mathcal{F}(M)\mathcal{F}(B) \text{ for arbitrary } \text{II}_1 \text{ factor } B.$$

In this case, we say that  $M$  satisfies the *tensor factorization property for fundamental groups* (say, *property (TFF)* in short).

Our first theorem treats examples of  $\text{II}_1$  factors with property (TFF). See [BO08, Definitions 12.3.1 and 15.1.2] for definitions of weak amenability and bi-exactness (and note that free groups, more generally hyperbolic groups, satisfy them).

**Theorem A.** *Let  $M$  be one of the following  $\text{II}_1$  factors.*

- *A group  $\text{II}_1$  factor  $L\Gamma$ , where  $\Gamma$  is an ICC, non-amenable, weakly amenable, and bi-exact group.*
- *A free product  $\text{II}_1$  factor  $M_1 * M_2$ , where  $M_1$  and  $M_2$  are diffuse (and tracial).*
- *A group  $\text{II}_1$  factor  $L(\Delta \wr \Lambda)$ , where  $\Delta$  is a non-trivial amenable group and  $\Lambda$  is a non-amenable group.*

*Then  $M$  satisfies the property (TFF).*

As a corollary of this theorem, we provide the main observation of this article.

**Corollary B.** *For any  $G \in \mathcal{S}_{\text{factor}}$ , there is a  $\text{II}_1$  factor  $M$  with separable predual and with the property (TFF) such that  $\mathcal{F}(M) = G$ .*

*The class  $\mathcal{S}_{\text{factor}}$  admits the following properties.*

- *Stability under multiplication: for any  $G, H \in \mathcal{S}_{\text{factor}}$ , the group  $G \cdot H$  is in  $\mathcal{S}_{\text{factor}}$ .*

- *Stability under countable intersection:* for any  $G_n \in \mathcal{S}_{\text{factor}}$  (possibly  $G_n = G_m$  for  $n \neq m$ ),  $n \in \mathbb{N}$ , the group  $\bigcap_n G_n$  is in  $\mathcal{S}_{\text{factor}}$ .

We note that the proof of the first statement in this corollary in fact shows the following: if we put  $N := L\mathbb{F}_n * L(\mathbb{Z}^2 \rtimes \text{SL}(2, \mathbb{Z}))$ , then for *arbitrary*  $\text{II}_1$  factor  $B$  we have

$$\mathcal{F}(B) = \mathcal{F}(*_{\mathbb{N}}(B \overline{\otimes} N)).$$

Thus the free product  $\text{II}_1$  factor  $*_{\mathbb{N}}(B \overline{\otimes} N)$  does the work. We also note that the second one in this corollary states general properties for the class  $\mathcal{S}_{\text{factor}}$ . Although it may not be useful to solve the aforementioned question by Murray and von Neumann, this is an interesting consequence since there are very few general properties for the class  $\mathcal{S}_{\text{factor}}$  as we mentioned.

The proof of Theorem A uses the idea in our previous paper [Is14], in which we introduced another notion of primeness for  $\text{II}_1$  factors. Recall that a  $\text{II}_1$  factor  $M$  is said to be *prime* if it does not have a tensor decomposition as  $\text{II}_1$  factors, namely, if it has a decomposition  $M = M_1 \overline{\otimes} M_2$ , then at least one  $M_i$  must be of type I. Obviously this definition comes from the notion of prime numbers in the number theory.

Actually there are two equivalent notions of prime numbers. Recall that a number  $p \in \mathbb{N}$  is *irreducible* if for any  $q, r \in \mathbb{N}$  with  $p = qr$ , we have  $q = 1$  or  $r = 1$ ; and is *prime* if for any  $q, r, s \in \mathbb{N}$  with  $pq = rs$ , we have  $p \mid r$  or  $p \mid s$ . In the von Neumann algebra theory, we adapt the first one (i.e. irreducibility) as a definition of primeness. In [Is14, Section 5], we introduced a different notion of primeness, which corresponds to the second one as follows. To distinguish two primeness, we name it *strongly prime*.

- We say a  $\text{II}_1$  factor  $M$  is *strongly prime* if for any  $\text{II}_1$  factors  $B, K$  and  $L$  with  $M \overline{\otimes} B = K \overline{\otimes} L$ , there is a unitary  $u \in \mathcal{U}(M \overline{\otimes} B)$  and  $t > 0$  such that, under the identification  $K \overline{\otimes} L = K^t \overline{\otimes} L^{1/t}$ , we have  $uMu^* \subset K^t$  or  $uMu^* \subset L^{1/t}$ .

Here we identify each tensor component as a subalgebra (e.g.  $M = M \otimes \mathbb{C} \subset M \overline{\otimes} B$ ).

Our second main theorem treats examples of strongly prime factors. Note that the first item in this theorem was already obtained in our previous article [Is14, Theorem 5.1]. We also note that the first and the second item in the theorem treat exactly the same ones as in Theorem A.

**Theorem C.** *Let  $M$  be one of the following  $\text{II}_1$  factors.*

- *A group  $\text{II}_1$  factor  $L\Gamma$ , where  $\Gamma$  is a non-amenable, ICC, weakly amenable, and bi-exact group.*
- *A free product  $\text{II}_1$  factor  $M_1 * M_2$ , where  $M_1$  and  $M_2$  are diffuse.*
- *A group  $\text{II}_1$  factor  $L(\Delta \wr \Lambda)$ , where  $\Delta$  is a non-trivial amenable group and  $\Lambda$  is a direct product of finitely many non-amenable, weakly amenable, and bi-exact groups.*

*Then  $M$  is strongly prime.*

In section 3, we will show that the property (TFF) has a sufficient condition similar to strong primeness (Lemma 3.4), and hence strong primeness is actually a sufficient condition to the property (TFF) (Proposition 3.6). We note that strong primeness implies primeness, but the converse fails (Propositions 3.6 and 3.7).

We will also discuss unique prime factorization result, using the strong primeness. This particularly provides the first example of unique prime factorization result for *infinite* tensor products. Below we say that a  $\text{II}_1$  factor  $M$  is *semiprime* if for any tensor decomposition  $M = M_1 \overline{\otimes} M_2$ , at least one  $M_i$  is amenable. The reason we use semiprimeness is that any infinite tensor product factor  $M$  is McDuff (i.e.  $M \simeq M \overline{\otimes} R$  for the AFD  $\text{II}_1$  factor  $R$ ), so tensor components are determined up to tensor product with  $R$ .

**Proposition D.** *Let  $m, n \in \mathbb{N} \cup \{\infty\}$ . Let  $M_i$  be strongly prime  $\text{II}_1$  factors, and any  $N_j$   $\text{II}_1$  factors such that  $M := \overline{\otimes}_{i=1}^m M_i = \overline{\otimes}_{j=1}^n N_j$ . Then there is a unique map  $\sigma: \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$  such that  $M_i \preceq_M N_{\sigma(i)}$  for all  $i \in \{1, 2, \dots, m\}$ . In this case, the following statements hold true.*

- *The map  $\sigma$  is surjective if and only if all  $N_j$  are non-amenable.*
- *The map  $\sigma$  is injective if and only if all  $N_{\sigma(i)}$  are semiprime.*

*Thus the map  $\sigma$  is bijective if all  $N_j$  are non-amenable and semiprime. In this case for each  $i \in \{1, 2, \dots, m\}$ ,  $N_{\sigma(i)}$  is isomorphic to  $M_i^{t_i} \overline{\otimes} P_i$  for some  $t_i > 0$  and some amenable factor  $P_i$ .*

In the proposition, if we assume all  $N_j$  are prime, then the map  $\sigma$  is bijective and  $M_i$  and  $N_{\sigma(i)}$  are stably isomorphic for all  $i$ . We note that the map  $\sigma$  in the proposition is surjective whenever  $m < \infty$ , since  $M = M_1 \overline{\otimes} \dots \overline{\otimes} M_m$  is full and so  $N_j$  can not be amenable.

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## 2 Preliminaries

In this article, all von Neumann algebras that we consider are assumed to be finite and  $\sigma$ -finite, namely, they admit faithful normal tracial states.

### Popa's intertwining technique

We recall Popa's intertwining theorem. This is the main tool in the deformation/rigidity theory.

**Theorem 2.1** ([Po01, Po03]). *Let  $M$  be a finite von Neumann algebra with trace  $\tau$ ,  $p$  and  $q$  projections in  $M$ ,  $A \subset pMp$  and  $B \subset qMq$  von Neumann subalgebras with  $\tau$ -preserving conditional expectations  $E_A$  and  $E_B$ . Then the following conditions are equivalent.*

- (i) *There exist non-zero projections  $e \in A$ ,  $f \in B$ , a unital normal  $*$ -homomorphism  $\theta: eAe \rightarrow fBf$ , and a partial isometry  $v \in eMf$  such that  $v\theta(x) = xv$  for all  $x \in eAe$ .*
- (i)' *There exist a nonzero normal  $*$ -homomorphism  $\psi: A \rightarrow B \overline{\otimes} \mathbb{M}_n$  for some  $n \in \mathbb{N}$  and a nonzero partial isometry  $w \in (p \otimes e_{1,1})(M \overline{\otimes} \mathbb{M}_n)$  such that  $w\psi(x) = (x \otimes e_{1,1})w$  for all  $x \in A$ , where  $(e_{i,j})_{i,j}$  is a fixed matrix unit in  $\mathbb{M}_n$ .*
- (ii) *There exists no net  $(w_i)_i$  of unitaries in  $A$  such that  $\|E_B(b^*w_i a)\|_{2,\tau} \rightarrow 0$  for any  $a, b \in pMq$ .*
- (iii) *There exists a positive element  $d \in p\langle M, B \rangle p \cap A'$  such that  $0 < \text{Tr}_{\langle M, B \rangle}(d) < \infty$ , where  $\text{Tr}_{\langle M, B \rangle}$  is the canonical trace on  $\langle M, B \rangle$  (with respect to  $\tau$ ).*

We write  $A \preceq_M B$  if one of these conditions holds.

Note that when  $B = \mathbb{C}$ ,  $A \not\preceq_M \mathbb{C}$  if and only if  $A$  is diffuse. We next observe some elementary lemmas.

**Lemma 2.2** ([HI15, Lemma 4.6]). *Let  $M$  and  $N$  be finite von Neumann algebras,  $p$  a projection in  $M$ ,  $A \subset M$ ,  $N_0 \subset N$  and  $B \subset M$  finite von Neumann subalgebras. Then  $A \preceq_M B$  if and only if  $A \otimes \mathbb{C}1_N \preceq_{M \overline{\otimes} N} B \otimes \mathbb{C}1_N$  if and only if  $A \otimes N_0 \preceq_{M \overline{\otimes} N} B \overline{\otimes} N$ .*

**Lemma 2.3.** *Let  $B$  be a finite von Neumann algebra and  $\Gamma$  a discrete group acting on  $B$  as a trace preserving action. Write  $M := B \rtimes \Gamma$ . Then  $M \preceq_M B$  if and only if  $L\Gamma \preceq_M B$  if and only if  $\Gamma$  is a finite group.*

**Proof.** If  $\Gamma$  is a finite group, then the canonical trace of the basic construction  $\langle M, B \rangle$  is finite. So by Theorem 2.1(iii), we get  $M \preceq_M B$ . If  $\Gamma$  is infinite, then we can find a sequence  $g_n \in \Gamma$  such that all  $g_n$  are distinct with each other. Then it satisfies Theorem 2.1(ii) and hence  $L\Gamma \not\preceq_M B$ . Finally by Theorem 2.1(i)', it is obvious that  $M \preceq_M B$  implies  $L\Gamma \preceq_M B$ .  $\square$

**Lemma 2.4.** *Let  $M = M_1 * M_2$  be a tracial free product von Neumann algebra,  $p \in M_1$  a projection, and let  $A \subset pM_1p$  be a diffuse von Neumann subalgebra. Then we have  $A \not\preceq_M M_2$ .*

**Proof.** We may assume  $M_2 \neq \mathbb{C}$ . Let  $(u_n)_n$  be a sequence of unitaries in  $A$  which converges to 0 weakly. By simple calculations, one can show that if each  $a, b \in M$  is a scalar or a reduced word, then  $\|E_{M_2}(b^*u_n a)\|_2$  converges to 0 as  $n \rightarrow \infty$ . Hence by Theorem 2.1 (ii),  $A \not\preceq_M M_2$  holds.  $\square$

In the lemma below, we denote the *normalizer* for an inclusion  $B \subset M$  by  $\mathcal{N}_M(B) := \{u \in \mathcal{U}(M) \mid uBu^* = B\}$ .

**Lemma 2.5.** *Let  $B \subset M$  be finite von Neumann algebras,  $p$  a projection in  $M$ , and  $A, P \subset pMp$  von Neumann subalgebras. Assume that  $A$  and  $P$  commute. Assume  $A \preceq_M B$  and  $P \preceq_M B$ . If  $B$  is regular (i.e.  $\mathcal{N}_M(B)'' = M$ ) and  $\mathcal{N}_{pMp}(A)' \cap pMp = \mathbb{C}p$ , then we have  $(A \vee P) \preceq_M B$ .*

**Proof.** We follow the proof of [Sa09, Lemma 33].

By Theorem 2.1 (iii), we find a positive element  $d_A \in A' \cap p\langle M, B \rangle p$  with  $0 < \text{Tr}_{\langle M, B \rangle}(d) < \infty$ . Taking a spectral projection, we may assume  $d_A$  is a projection. Observe that for any  $u \in \mathcal{N}_{pMp}(A)$  and  $v \in \mathcal{N}_M(B)$ , the element  $v^{\text{op}}ud_Au^*(v^{\text{op}})^*$  satisfies the same condition as the one on  $d_A$ , where  $v^{\text{op}}$  is the right action of  $v$  on  $L^2(M)$  (we indeed have  $\text{Tr}_{\langle M, B \rangle} \circ \text{Ad } v^{\text{op}} = \text{Tr}_{\langle M, B \rangle}$ , since  $(v^{\text{op}})^*ve_Bv^{\text{op}}v^* = e_B$ . See [BO08, Exercise F.6] for the construction of  $\text{Tr}_{\langle M, B \rangle}$  and use the fact that  $\text{Tr}_{\langle M, B \rangle}$  is uniquely determined by  $\text{Tr}_{\langle M, B \rangle}(x^*e_Bx) = \tau(x^*x)$  for  $x \in M$ ). So the element  $d := \sup\{v^{\text{op}}ud_Au^*(v^{\text{op}})^* \mid u \in \mathcal{N}_{pMp}(A), v \in \mathcal{N}_M(B)\}$  is contained in

$$\begin{aligned} & A' \cap \mathcal{N}_{pMp}(A)' \cap p\langle M, B \rangle p \cap p(\mathcal{N}_M(B)^{\text{op}})'p \\ &= \mathcal{N}_{pMp}(A)' \cap p\langle M, B \rangle p \cap pMp \\ &= \mathcal{N}_{pMp}(A)' \cap pMp = \mathbb{C}p. \end{aligned}$$

Hence we get  $d = p$ . Let now  $d_P$  be a non-zero trace finite projection in  $P' \cap p\langle M, B \rangle p$ . Then since  $d = p$ , there are finite subsets  $\mathcal{E} \subset \mathcal{N}_{pMp}(A)$  and  $\mathcal{F} \subset \mathcal{N}_M(B)$  satisfying that  $\bigvee_{u \in \mathcal{E}, v \in \mathcal{F}} v^{\text{op}}ud_Au^*(v^{\text{op}})^*$  is not orthogonal to  $d_P$ . Thus up to exchanging  $d_A$  with this element, we can assume  $d_Ad_P \neq 0$ .

Consider a convex subset  $K := \overline{\text{co}}^w\{ud_Au^* \mid u \in \mathcal{U}(P)\} \subset \langle M, B \rangle$  and observe that  $K$  is regarded as a subset in  $L^2(\langle M, B \rangle, \text{Tr}_{\langle M, B \rangle})$  which is  $L^2$ -norm bounded (e.g. [BO08, Exercise F.3]). Take the unique minimal  $L^2$ -norm element  $\tilde{d}$  in  $K$ . We have  $u\tilde{d}u^* = \tilde{d}$  for any  $u \in \mathcal{U}(P)$  by the uniqueness, and hence  $\tilde{d}$  is contained in  $P' \cap A' \cap p\langle M, B \rangle p =$

$(A \vee P)' \cap p\langle M, B \rangle p$ . Observe that  $\tilde{d}$  is trace finite in  $\langle M, B \rangle$  since so is  $d_A$  (and  $\text{Tr}_{\langle M, B \rangle}$  is normal). Finally  $\tilde{d}$  is non-zero since for any  $u \in \mathcal{U}(P)$ ,

$$\langle ud_A u^*, d_P \rangle = \text{Tr}_{\langle M, B \rangle}(ud_A u^* d_P) = \text{Tr}_{\langle M, B \rangle}(d_A d_P) > 0,$$

and so any  $a \in K$  satisfies  $\langle a, d_P \rangle = \text{Tr}_{\langle M, B \rangle}(d_A d_P) > 0$ . Thus we obtain  $(A \vee P) \preceq_M B$ .  $\square$

### Relative amenability

We next recall relative amenability introduced in [OP07].

**Definition 2.6** ([OP07, Definition 2.2]). Let  $M$  be a finite von Neumann algebra with trace  $\tau$ . Let  $p \in M$  be a projection and  $A \subset pMp$  and  $B \subset M$  von Neumann subalgebras. We say  $A$  is *amenable relative to  $B$  in  $M$* , and write as  $A \triangleleft_M B$ , if there exists a conditional expectation from  $p\langle M, B \rangle p$  onto  $A$  which restricts to a  $\tau$ -preserving expectation on  $pMp$ .

**Proposition 2.7** ([OP07, Proposition 2.4(3)]). Let  $B \subset M$  and  $A \subset pMp$  as above, and let  $N \subset M$  be another von Neumann subalgebra. If  $A \triangleleft_M B$  and  $B \triangleleft_M N$ , then  $A \triangleleft_M N$ .

We record the following elementary lemma.

**Lemma 2.8.** Let  $M$  and  $B$  be finite von Neumann algebras. Then  $M \overline{\otimes} B \triangleleft_{M \overline{\otimes} B} B$  if and only if  $M$  is amenable.

## 3 Some observations on tensor product $\text{II}_1$ factors

In this section, we briefly review fundamental properties of tensor product  $\text{II}_1$  factors to study the property (TFF) and strong primeness. We say  $M_1 \overline{\otimes} \cdots \overline{\otimes} M_m = N_1 \overline{\otimes} \cdots \overline{\otimes} N_n$  is a *tensor decomposition as  $\text{II}_1$  factors* if each  $M_i$  and  $N_j$  is a  $\text{II}_1$  factor.

### Property (TFF)

We first recall the following observation of Ozawa and Popa. This shows that, to see a unitary embedding on tensor products, we have only to find Popa's conjugacy “ $\preceq$ ” introduced in Theorem 2.1. This allows us to reformulate strong primeness (Lemma 3.5), so that we can make use of results in the deformation/rigidity theory.

**Lemma 3.1** ([OP03, Proposition 12]). Let  $M_1 \overline{\otimes} M_2 = N_1 \overline{\otimes} N_2$  ( $=: M$ ) be a tensor decomposition as  $\text{II}_1$  factors. Then  $N_1 \preceq_M M_1$  if and only if there is a unitary element  $u \in M$  and a decomposition  $M = M_1^t \overline{\otimes} M_2^{1/t}$  for some  $t > 0$  such that  $uN_1u^* \subset M_1^t$ .

**Remark 3.2.** In this lemma, we are having an identification  $M = M_1^t \overline{\otimes} M_2^{1/t}$ , using a non-canonical isomorphism  $M_1 \overline{\otimes} M_2 \simeq M_1^t \overline{\otimes} M_2^{1/t}$ . Since this isomorphism is given at the level of a partial isometry conjugacy of  $M_1 \overline{\otimes} M_2 \overline{\otimes} \mathbb{M}_n$  (for some large  $n \in \mathbb{N}$ ), one can show that  $N_1 \preceq_M M_1$  if and only if  $N_1 \preceq_M M_1^t$  for any  $t > 0$  and any such an identification  $M_1 \overline{\otimes} M_2 = M_1^t \overline{\otimes} M_2^{1/t}$ . So we do not need to be careful to identify  $M_1 \overline{\otimes} M_2$  with  $M_1^t \overline{\otimes} M_2^{1/t}$  in the study of Popa's conjugacy.

Here we record a simple but very useful lemma on tensor product factors.

**Lemma 3.3.** Let  $M_1 \overline{\otimes} M_2 = N_1 \overline{\otimes} N_2$  ( $=: M$ ) be a tensor decomposition as  $\text{II}_1$  factors and assume that  $M_1 \subset N_1$ . Then  $M'_1 \cap N_1$  is a factor and satisfies

$$M_2 = (M'_1 \cap N_1) \overline{\otimes} N_2 \quad \text{and} \quad N_1 = M_1 \overline{\otimes} (M'_1 \cap N_1).$$

**Proof.** Since  $M_1 \subset N_1$ , we have

$$M_2 = M'_1 \cap M = M'_1 \cap (N_1 \overline{\otimes} N_2) = (M'_1 \cap N_1) \overline{\otimes} N_2.$$

So  $M'_1 \cap N_1$  is a factor. We have  $M = M_1 \overline{\otimes} (M'_1 \cap N_1) \overline{\otimes} N_2$  and hence  $N_1 = N'_2 \cap M = M_1 \overline{\otimes} (M'_1 \cap N_1)$ .  $\square$

The following lemma is a key observation in this paper, which states a sufficient condition to the property (TFF) in terms of Popa's conjugacy. Although its proof is easy, this lemma plays significant roles in our study.

**Lemma 3.4.** *Let  $M$  be a prime  $\text{II}_1$  factor satisfying the following condition.*

- *For any  $\text{II}_1$  factor  $B$  and any  $t > 0$  such that  $M \overline{\otimes} B \simeq M \overline{\otimes} B^t (= K \overline{\otimes} L)$ , under the identification  $M \overline{\otimes} B = K \overline{\otimes} L$ , we have either*

$$K \preceq_{M \overline{\otimes} B} B, \quad L \preceq_{M \overline{\otimes} B} B, \quad M \preceq_{M \overline{\otimes} B} L, \quad \text{or} \quad B \preceq_{M \overline{\otimes} B} L.$$

*Then  $M$  has the property (TFF).*

**Proof.** Fix a  $\text{II}_1$  factor  $B$  and take  $t \in \mathcal{F}(M \overline{\otimes} B)$ . We will show  $t \in \mathcal{F}(M)\mathcal{F}(B)$ . Fix an isomorphism  $M \overline{\otimes} B \simeq (M \overline{\otimes} B)^t \simeq M \overline{\otimes} B^t (= K \overline{\otimes} L)$ . By assumption, regarding  $N := M \overline{\otimes} B^t = K \overline{\otimes} L$ , we have either

$$K \preceq_N B, \quad L \preceq_N B, \quad M \preceq_N L, \quad \text{or} \quad B \preceq_N L.$$

If  $L \preceq_N B$ , then by Lemma 3.1 there exists  $s > 0$  and  $u \in \mathcal{U}(N)$  such that  $uLu^* \subset B^s$  under the isomorphism  $M \overline{\otimes} B = M^{1/s} \overline{\otimes} B^s$ . For simplicity we assume  $u = 1$ . Then by Lemma 3.3, putting  $P := L' \cap B^s$ , it holds that

$$B^s = L \overline{\otimes} P \quad \text{and} \quad K = P \overline{\otimes} M^{1/s}.$$

Since  $K(= M)$  is prime,  $P$  is finite dimensional. Write  $P = \mathbb{M}_n$  for some  $n \in \mathbb{N}$  and we obtain

$$B^s = L^n \quad \text{and} \quad K = M^{n/s}.$$

Since  $L^n = B^{tn}$  and  $K = M$ , this implies that  $s/tn \in \mathcal{F}(B)$  and  $n/s \in \mathcal{F}(M)$ , and hence

$$\frac{1}{t} = \frac{s}{tn} \cdot \frac{n}{s} \in \mathcal{F}(B)\mathcal{F}(M).$$

Thus  $t \in \mathcal{F}(B)\mathcal{F}(M)$ .

Next assume  $K \preceq_N B$ . Then by the same reasoning as above, there exists  $s > 0$  and  $u \in \mathcal{U}(N)$  such that  $uKu^* \subset B^s$ . We assume  $u = 1$ . Putting  $Q := K' \cap B^s$  it holds that

$$B^s = K \overline{\otimes} Q \quad \text{and} \quad L = Q \overline{\otimes} M^{1/s}.$$

Since  $K = M$  and  $L = B^t$ , these equations imply

$$B^s = M \overline{\otimes} Q \quad \text{and} \quad B^t = Q \overline{\otimes} M^{1/s},$$

and hence

$$B^s = M \overline{\otimes} Q \simeq Q \overline{\otimes} M = B^{ts}.$$

This implies  $t \in \mathcal{F}(B)$  and we obtain the conclusion.

Finally assume that  $M \preceq_N L$  or  $B \preceq_N L$ . Then since  $K = M$  and  $L = B^t$ , if we put  $\tilde{B} := B^t$ ,  $\tilde{t} := 1/t$ ,  $\tilde{K} := M$ ,  $\tilde{L} := \tilde{B}^t$ , and  $\tilde{K} \overline{\otimes} \tilde{L} = M \overline{\otimes} \tilde{B}$ , we can apply exactly the same argument as in the previous two cases, and obtain  $\tilde{t} = 1/t \in \mathcal{F}(M)\mathcal{F}(\tilde{B})$ . Since  $\mathcal{F}(\tilde{B}) = \mathcal{F}(B^t) = \mathcal{F}(B)$ , we obtain the conclusion.  $\square$

## Strong primeness

We study fundamental properties on strong primeness. We first give a reformulation of strong primeness in terms of Popa's conjugacy.

**Lemma 3.5.** *A  $\text{II}_1$  factor  $M$  is strongly prime if and only if for any tensor decomposition  $M \overline{\otimes} B = K \overline{\otimes} L$  as  $\text{II}_1$  factors, we have either  $K \preceq B$  or  $L \preceq B$ .*

**Proof.** Use Lemma 3.1 and [Va08, Lemma 3.5].  $\square$

We deduce primeness from strong primeness. This is not entirely trivial since, in the definition of strong primeness, we mention only a decomposition as  $\text{II}_1$  factors.

**Proposition 3.6.** *Strong primness implies primeness. In particular any strongly prime  $\text{II}_1$  factor satisfies the property (TFF).*

**Proof.** Let  $M$  be a non-prime  $\text{II}_1$  factor with a decomposition  $M = M_1 \overline{\otimes} M_2$  as  $\text{II}_1$  factors. Fix any  $\text{II}_1$  factor  $B$  and put  $K := M_1$ ,  $L := M_2 \overline{\otimes} B$ , and  $N := M \overline{\otimes} B = K \overline{\otimes} L$ . Then if  $M$  is strongly prime, we have either  $M \preceq_N K$  or  $M \preceq_N L$ . By Lemma 2.2, the first one is equivalent to  $M_2 \preceq_{M_2} \mathbb{C}$  and the second one is to  $M_1 \preceq_{M_1} \mathbb{C}$ . Thus in each case, we get a contradiction. Use Lemmas 3.4 and 3.5 for the second assertion.  $\square$

We observe the difference of two primeness. This follows from [Ho15, Theorem B].

**Proposition 3.7.** *Any strongly prime  $\text{II}_1$  factor is full. In particular there is a prime  $\text{II}_1$  factor, which is not strongly prime.*

**Proof.** Let  $M$  and  $B$  be non-full  $\text{II}_1$  factors. Then by [Ho15, Theorem B], there is an automorphism  $\phi$  on  $M \overline{\otimes} B$  such that  $\phi(M) \not\preceq_{M \overline{\otimes} B} B$  and  $\phi(B) \not\preceq_{M \overline{\otimes} B} B$ . Thus the decomposition  $M \overline{\otimes} B = \phi(M) \overline{\otimes} \phi(B)$  shows that  $M$  is not strongly prime.

Let  $\mathbb{F}_2 \curvearrowright X$  be a free, ergodic, and measure preserving action of the free group on a standard probability space. Assume that it is not strongly ergodic. Then the crossed product  $M := L^\infty(X) \rtimes \mathbb{F}_2$  is a prime  $\text{II}_1$  factor by [Oz04, Theorem 4.6], and is not strongly prime since it is not full.  $\square$

## 4 Proof of Proposition D

We study a *unique prime factorization* phenomena, by using our strong primeness. This was already mentioned in our previous paper [Is14, Corollary 5.1.3], that shows strongly prime factors behave like prime numbers with respect to von Neumann algebra tensor products. We only discussed the case of tensor products with finitely many strongly prime factors. So in this paper, we study the case of infinite tensor products.

We start with several lemmas.

**Lemma 4.1.** *Let  $M$  be a strongly prime  $\text{II}_1$  factor and let  $M \overline{\otimes} B = N_1 \overline{\otimes} \cdots \overline{\otimes} N_n$  ( $=: N$ ) be a tensor decomposition as  $\text{II}_1$  factors with  $n \geq 2$ . Then there is  $i$  such that  $M \preceq_N N_i$ .*

**Proof.** We prove it by induction on  $n$ . The case  $n = 2$  is obvious by the definition of strong primeness. So assuming  $n - 1 \geq 2$  is proven, we show the case  $n$  holds.

Put  $N'_1 := N'_1 \cap N$ . Then since  $M \overline{\otimes} B = N_1 \overline{\otimes} N'_1$ , we have either  $M \preceq_N N_1$  or  $M \preceq_N N'_1$ . Since  $M \preceq_N N_1$  implies the conclusion, we may assume  $M \preceq_N N'_1$ . By Lemma 3.1 we find  $u \in \mathcal{U}(N)$  and  $t > 0$  such that  $uMu^* \subset (N'_1)^t$ . Then by Lemma 3.3 we have  $uMu^* \overline{\otimes} P = (N'_1)^t$ , where  $P = (uMu^*)' \cap (N'_1)^t$ . Observe that  $P$  is a  $\text{II}_1$  factor. In fact, if  $P$  is finite dimensional, then because  $M$  is prime,  $n$  must be 2 which contradicts our assumption.



Now we can apply strong primeness of  $M$  and the assumption on the induction to the decomposition  $uMu^* \overline{\otimes} P = N_2 \overline{\otimes} \cdots \overline{\otimes} N_n$  and get that  $uMu^* \preceq_{uMu^* \overline{\otimes} P} N_i$  for some  $i \geq 2$ . Then take  $\theta, p, q, v$  as in Theorem 2.1(i), and observe that  $\theta \circ \text{Ad } u, u^*pu, q, u^*v$  gives the condition  $M \preceq_N N_i$ . Thus we get the conclusion.  $\square$

**Lemma 4.2.** *Let  $M \overline{\otimes} B = N_1 \overline{\otimes} \cdots \overline{\otimes} N_n (=: N)$  be a tensor decomposition as  $\text{II}_1$  factors with  $n \geq 2$ . If  $M \preceq_N N_i$  and  $M \preceq_N N_j$ , then  $i = j$ .*

**Proof.** Suppose by contradiction that  $i \neq j$ , and put  $i = 1$  and  $j = 2$  for simplicity. Then by Lemmas 3.1 and 3.3, one has  $uMu^* \overline{\otimes} P = N_1^t$ , where  $u \in \mathcal{U}(N)$ ,  $t > 0$ , and  $P := (uMu^*)' \cap N_1^t$ , that gives a decomposition

$$N = uMu^* \overline{\otimes} P \overline{\otimes} N_2^{1/t} \overline{\otimes} N_3 \overline{\otimes} \cdots \overline{\otimes} N_n.$$

Observe by Lemma 3.1 that the given condition  $M \preceq_N N_2$  is equivalent to  $uMu^* \preceq_N N_2^{1/t}$ . By Lemma 2.2, we get  $uMu^* \preceq_{N \cap (N_2^{1/t})'} \mathbb{C}$ , which contradicts the diffuseness of  $M$ .  $\square$

**Lemma 4.3.** *Let  $M_1 \overline{\otimes} M_2 \overline{\otimes} B = N_1 \overline{\otimes} \cdots \overline{\otimes} N_n (=: N)$  be a tensor decomposition as  $\text{II}_1$  factors with  $n \geq 2$ . If  $M_1 \preceq_N N_1$  and  $M_2 \preceq_N N_1$ , then  $M_1 \overline{\otimes} M_2 \preceq_N N_1$ . In this case,  $N_1$  is not prime.*

**Proof.** The first assertion is immediate by Lemma 2.5. For the second one, by Lemmas 3.1 and 3.3, take  $u \in \mathcal{U}(N)$  and  $t > 0$  such that  $u(M_1 \overline{\otimes} M_2)u^* \subset N_1^t$  and  $N_1^t = u(M_1 \overline{\otimes} M_2)u^* \overline{\otimes} P$ , where  $P := u(M_1 \overline{\otimes} M_2)'u^* \cap N_1^t$ . Since  $N_1 \simeq M_1^{1/t} \overline{\otimes} M_2 \overline{\otimes} P$ ,  $N_1$  is not prime.  $\square$

**Lemma 4.4.** *Let  $\overline{\otimes}_{i=1}^m M_i = N \overline{\otimes} B (=: M)$  be a tensor decomposition as  $\text{II}_1$  factors with  $m = \infty$ . If  $B \preceq_M \overline{\otimes}_{i=k}^m M_i$  for all  $k \in \mathbb{N}$ , then  $B$  is amenable.*

**Proof.** We follow the idea in [HU15, Proposition 4.2] due to Ioana. In the proof, for any subset  $\mathcal{F} \subset \mathbb{N}$  we put  $M_{\mathcal{F}} := \overline{\otimes}_{i \in \mathcal{F}} M_i \subset M$ .

Put  $\mathcal{M} := M \overline{\otimes} M$  and we regard the left  $M$  as the original one. Let  $\Sigma$  be the flip map on  $\mathcal{M}$  given by  $\Sigma(a \otimes b) = b \otimes a$ . For any  $\mathcal{F} \subset \mathbb{N}$ , put  $\mathcal{M}_{\mathcal{F}} := M_{\mathcal{F}} \overline{\otimes} M_{\mathcal{F}}$  with the flip  $\Sigma_{\mathcal{F}}$ . We regard  $\Sigma_{\mathcal{F}} \in \text{Aut}(\mathcal{M})$  by putting  $\Sigma_{\mathcal{F}}|_{\mathcal{M}_{\mathcal{F}^c}} := \text{id}$ . Observe that  $\text{weak-}\lim_{\mathcal{F}} \Sigma_{\mathcal{F}}(x) = \Sigma(x)$  for all  $x \in \mathcal{M}$ , where the limit is taken over all *finite* subsets  $\mathcal{F} \subset \mathbb{N}$ .

Observe next  $B \preceq_M M_{\mathcal{F}^c}$  for any finite  $\mathcal{F} \subset \mathbb{N}$  by assumption, so there is a unitary  $v_{\mathcal{F}} \in M$  and  $t_{\mathcal{F}} > 0$  such that  $v_{\mathcal{F}} B v_{\mathcal{F}}^* \subset M_{\mathcal{F}^c}^{t_{\mathcal{F}}}$  by Lemma 3.1. In this case, we may assume that  $M_{\mathcal{F}^c}^{t_{\mathcal{F}}} \subset M_{\max \mathcal{F}} \overline{\otimes} M_{\mathcal{F}^c}$  where  $\max \mathcal{F} := \max\{i \mid i \in \mathcal{F}\}$  (recall that we are fixing  $M = M_{\mathcal{F}}^{1/t_{\mathcal{F}}} \overline{\otimes} M_{\mathcal{F}^c}^{t_{\mathcal{F}}}$ , so applying again a partial isometry conjugacy at the level of  $M \overline{\otimes} M_n$  for some  $n \in \mathbb{N}$  we may assume this condition). In particular we have  $\Sigma_{\mathcal{F}^c}(v_{\mathcal{F}} b v_{\mathcal{F}}^* \otimes 1) = v_{\mathcal{F}} b v_{\mathcal{F}}^* \otimes 1$  for all  $b \in B$ , where  $\mathcal{F}^c := \mathbb{N} \setminus \max \mathcal{F}$ . We put  $u_{\mathcal{F}} := (v_{\mathcal{F}}^* \otimes 1) \Sigma_{\mathcal{F}^c}(v_{\mathcal{F}} \otimes 1) \in \mathcal{U}(\mathcal{M})$  and calculate that for all  $b \in B$ ,

$$\begin{aligned} u_{\mathcal{F}} \Sigma_{\mathcal{F}^c}(b \otimes 1) &= (v_{\mathcal{F}}^* \otimes 1) \Sigma_{\mathcal{F}^c}(v_{\mathcal{F}} \otimes 1) \Sigma_{\mathcal{F}^c}(b \otimes 1) \\ &= (v_{\mathcal{F}}^* \otimes 1) \Sigma_{\mathcal{F}^c}(v_{\mathcal{F}} b \otimes 1) \\ &= (v_{\mathcal{F}}^* \otimes 1) \Sigma_{\mathcal{F}^c}(v_{\mathcal{F}} b v_{\mathcal{F}}^* \otimes 1) \Sigma_{\mathcal{F}^c}(v_{\mathcal{F}} \otimes 1) \\ &= (v_{\mathcal{F}}^* \otimes 1) (v_{\mathcal{F}} b v_{\mathcal{F}}^* \otimes 1) \Sigma_{\mathcal{F}^c}(v_{\mathcal{F}} \otimes 1) \\ &= (b \otimes 1) u_{\mathcal{F}}. \end{aligned}$$

Define a state  $\Omega$  on  $\mathbb{B}(L^2(\mathcal{M}))$  by  $\Omega(X) := \lim_{\mathcal{F}} \langle X u_{\mathcal{F}}, u_{\mathcal{F}} \rangle_{L^2(\mathcal{M})}$ , where the limit is taken over all finite  $\mathcal{F}$ . It satisfies for  $x \in \mathcal{M}$

$$\Omega(a) = \lim_{\mathcal{F}} \langle a u_{\mathcal{F}}, u_{\mathcal{F}} \rangle_{L^2(\mathcal{M})} = \lim_{\mathcal{F}} \tau_{\mathcal{M}}(u_{\mathcal{F}}^* a u_{\mathcal{F}}) = \tau_{\mathcal{M}}(a).$$

For all  $b \in \mathcal{U}(B)$ , regarding  $L^2(\mathcal{M}) = L^2(M) \otimes L^2(M)$  with the right  $M$ -action given by  $M \ni x \mapsto 1 \otimes J_M x^* J_M$  where  $J_M$  is the anti-unitary map  $J_M(y) = y^*$  for  $y \in M \subset L^2(M)$ , since  $u_{\mathcal{F}} \Sigma_{\mathcal{F}^\circ}(b \otimes 1) = (b \otimes 1) u_{\mathcal{F}}$  and  $\Sigma_{\mathcal{F}^\circ}(b \otimes 1) \rightarrow \Sigma(b \otimes 1)$  weakly for all  $b \in B$ , we have

$$\begin{aligned} \Omega(b \otimes J_M b J_M) &= \lim_{\mathcal{F}} \langle (b \otimes J_M b J_M) u_{\mathcal{F}}, u_{\mathcal{F}} \rangle_{L^2(\mathcal{M})} \\ &= \lim_{\mathcal{F}} \langle (b \otimes 1) u_{\mathcal{F}} (1 \otimes b^*), u_{\mathcal{F}} \rangle_{L^2(\mathcal{M})} \\ &= \lim_{\mathcal{F}} \langle u_{\mathcal{F}} \Sigma_{\mathcal{F}^\circ}(b \otimes 1) (1 \otimes b^*), u_{\mathcal{F}} \rangle_{L^2(\mathcal{M})} \\ &= \lim_{\mathcal{F}} \tau_{\mathcal{M}}(\Sigma_{\mathcal{F}^\circ}(b \otimes 1) (1 \otimes b^*)) \\ &= \tau_{\mathcal{M}}(\Sigma(b \otimes 1) (1 \otimes b^*)) = 1. \end{aligned}$$

So the state  $\Omega$  satisfies  $\Omega((b \otimes J_M b J_M)(X \otimes 1)) = \Omega(X \otimes 1)$  and hence  $\Omega(b X b^* \otimes 1) = \Omega((b \otimes J_M b J_M)(X \otimes 1)(b \otimes J_M b J_M)^*) = \Omega(X)$  for all  $X \in \mathbb{B}(L^2(M))$  and  $b \in \mathcal{U}(B)$ . Thus the restriction of  $\Omega$  on  $\mathbb{B}(L^2(M)) \otimes \mathbb{C}1_{L^2(M)}$  is a  $B$ -central state which is the trace on  $B$ . This means  $B$  is amenable.  $\square$

*Proof of Proposition D.* We fix  $i \in \mathbb{N}$  with  $1 \leq i \leq m$ . Then since  $M_i$  is non-amenable, strong primeness and Lemma 4.4 imply that there is  $k \in \mathbb{N}$  with  $1 \leq k \leq n$  such that  $M_i \preceq_M \overline{\otimes}_{j=1}^k N_j$  (this is obvious if  $n \neq \infty$ ). By Lemmas 3.1 and 3.3, one has  $u M_i u^* \overline{\otimes} P = N_1^t \overline{\otimes} N_2 \overline{\otimes} \cdots \overline{\otimes} N_k$  for a factor  $P$ ,  $u \in \mathcal{U}(M)$ , and  $t > 0$ . Then if  $P$  is of type I, then  $k = 1$  by the primeness of  $M_i$  and hence  $M_i \preceq_M N_1$ . If  $P$  is a  $\text{II}_1$  factor, then by Lemma 4.1 there is some  $j \in \mathbb{N}$  with  $1 \leq j \leq k$  such that  $M_i \preceq_M N_j$ . Thus in any case there is  $j$  such that  $M_i \preceq_M N_j$ . We put  $\sigma(i) := j$ , and  $\sigma$  is uniquely determined by Lemma 4.2.

### Surjectivity of $\sigma$ .

Assume that  $\sigma$  is surjective. Then since the condition  $M_i \preceq_M N_{\sigma(i)}$  implies non-amenability of  $N_{\sigma(i)}$ , we have that all  $N_j$  are non-amenable.

To see the converse direction, we show the following claim.

**Claim.** Assume that there is  $j_0 \in \mathbb{N}$  with  $1 \leq j_0 \leq n$  such that  $M_i \not\preceq_M N_{j_0}$  for all  $i \in \mathbb{N}$  with  $1 \leq i \leq n$ . Then we have (i) a contradiction if  $m \neq \infty$ , and (ii)  $N_{j_0}$  is amenable if  $m = \infty$ .

**Proof.** We fix  $k \in \mathbb{N}$  with  $1 \leq k \leq m$ . Observe that  $M_i \preceq_M \overline{\otimes}_{j=1}^k N_{\sigma(j)}$  for all  $1 \leq i \leq k$ , and then Lemma 4.3 implies  $\overline{\otimes}_{i=1}^k M_i \preceq_M \overline{\otimes}_{j=1}^k N_{\sigma(j)}$ . By taking relative commutants, we have

$$N_{j_0} \subset (\overline{\otimes}_{j=1}^k N_{\sigma(j)})' \cap M \preceq_M (\overline{\otimes}_{i=1}^k M_i)' \cap M = \overline{\otimes}_{i=k+1}^m M_i.$$

If  $m \neq \infty$ , one can put  $k = m$  and obtain  $N_{j_0} \preceq_M \mathbb{C}$ , a contradiction. If  $m = \infty$ , then we have  $N_{j_0} \preceq_M \overline{\otimes}_{i=k+1}^m M_i$  for all  $k \in \mathbb{N}$  that implies amenability of  $N_{j_0}$  by Lemma 4.4.  $\square$

Observe now that  $j_0 \notin \text{Im} \sigma$  if and only if  $M_i \not\preceq_M N_{j_0}$  for all  $i \in \mathbb{N}$  with  $1 \leq i \leq n$  (since  $M_i \preceq_M N_{j_0}$  exactly means  $\sigma(i) = j_0$  by the uniqueness of  $\sigma$ ). So this claim shows that (i)  $\sigma$  is always surjective if  $m \neq \infty$ , and (ii) if  $m = \infty$  non-surjectivity of  $\sigma$  implies amenability of  $N_{j_0}$  for some  $j_0$ . This completes the statement for surjectivity (note that  $N_j$  can not be amenable if  $m \neq \infty$  as we mentioned in Introduction).

### Injectivity of $\sigma$ .

Assume next that  $\sigma$  is not injective. Then there are  $i \neq i'$  such that  $\sigma(i) = \sigma(i') =: j$ , that means  $M_i \preceq_M N_j$  and  $M_{i'} \preceq_M N_j$ . By (the proof of) Lemma 4.3,  $N_j$  is isomorphic to  $M_i^t \overline{\otimes} M_{i'} \overline{\otimes} P$  for some  $t > 0$  and a factor  $P$ . Since  $M_i$  and  $M_{i'}$  are non-amenable,  $N_j$  is not semiprime.

Conversely assume  $N_{\sigma(i)}$  is not semiprime for some  $i$ , so there is a tensor decomposition  $N_{j_0} = N_{j_0}^1 \overline{\otimes} N_{j_0}^2$  with non-amenable  $\text{II}_1$  factors  $N_{j_0}^1$  and  $N_{j_0}^2$ . If  $M_i \not\preceq_M N_{j_0}^1$  for all  $i$ , then the claim above shows that we have (i) a contradiction if  $m \neq \infty$ , and (ii)  $N_{j_0}^1$  is amenable if  $m = \infty$ . So non-amenable of  $N_{j_0}^1$  and  $N_{j_0}^2$  implies there is  $k, l \in \mathbb{N}$  such that  $M_k \preceq_M N_{j_0}^1$  and  $M_l \preceq_M N_{j_0}^2$ . We know  $k \neq l$  by Lemma 4.2. Finally since  $N_{j_0}^1, N_{j_0}^2 \subset N_{j_0}$ , we have  $M_k \preceq_M N_{j_0}$  and  $M_l \preceq_M N_{j_0}$  that means  $\sigma(k) = \sigma(l)$ . So  $\sigma$  is not injective.

Finally we assume that each  $N_j$  is non-amenable and semiprime. Then  $\sigma$  is bijective by previous arguments. By Lemmas 3.1 and 3.3,  $M_i \preceq_M N_{\sigma(i)}$  implies  $N_{\sigma(i)} \simeq M_i^{t_i} \overline{\otimes} P_i$  for some  $t_i > 0$  and a factor  $P_i$ . Since  $N_{\sigma(i)}$  is semiprime,  $P_i$  must be amenable.  $\square$

## 5 Proofs of main theorems

In the proofs of main theorems, we will make use of the following three structural theorems. Note that all of them are formulated with relative amenability, and this relativity is crucial to our proofs.

**Theorem 5.1** ([PV12, Theorem 1.4]). *Let  $B$  be any finite von Neumann algebra and  $\Gamma$  be weakly amenable and bi-exact group acting on  $B$ . Put  $M := B \rtimes \Gamma$ . Then for any von Neumann subalgebra  $A \subset M$  with amenable relative to  $B$  in  $M$ , we have either (i)  $A \preceq_M B$  or (ii)  $\mathcal{N}_M(A)''$  is amenable relative to  $B$  in  $M$ .*

**Theorem 5.2** ([Io12, Theorem 1.6][Va13, Theorem A]). *Let  $M = M_1 *_B M_2$  be an amalgamated free product of tracial von Neumann algebras  $(M_i, \tau)$  with common von Neumann subalgebra  $B \subset M_i$  w.r.t. the unique trace preserving conditional expectations. Let  $A \subset M$  be a von Neumann subalgebra that is amenable relative to  $B$  inside  $M$  and satisfies  $A \not\preceq_M B$ . Then we have either (i)  $\mathcal{N}_M(A)'' \preceq_M M_i$  for some  $i$  or (ii)  $\mathcal{N}_M(A)''$  is amenable relative to  $B$  inside  $M$ .*

**Theorem 5.3** ([SW11, Theorem 2.2]). *Let  $\Gamma$  be a wreath product group of a non-trivial amenable group by a non-amenable group and let  $B$  be a finite von Neumann algebra. Put  $M := B \overline{\otimes} L\Gamma$ . Let  $Q \subset M$  be a von Neumann subalgebra which is not amenable relative to  $B$ . If  $Q' \cap M$  is a regular subfactor in  $M$ , then we have  $Q' \cap M \preceq_M B$ .*

*Proof of Theorem C.* The first case was already proved in [Is14, Theorem 5.1.1].

Suppose by contradiction that  $M$  is not strongly prime. Then by Lemma 3.5, there are  $\text{II}_1$  factors  $B, K$ , and  $L$  such that  $B \overline{\otimes} M = K \overline{\otimes} L (= N)$  with  $K \not\preceq_N B$  and  $L \not\preceq_N B$ .

### Case 1. $M$ is a free product $M_1 * M_2$ .

By [BO08, Corollary F.14], there is a diffuse abelian subalgebra  $A \subset K$  such that  $A \not\preceq_N B$ . Then regarding  $N = (B \overline{\otimes} M_1) *_B (B \overline{\otimes} M_2)$ , we apply Theorem 5.2 to  $A \subset N$  and get either (i)  $\mathcal{N}_N(A)'' \preceq_N (B \overline{\otimes} M_i)$  for some  $i$  or (ii)  $\mathcal{N}_N(A)''$  is amenable relative to  $B$  in  $N$ .

Assume first that (ii) happens. Since  $L \subset \mathcal{N}_N(A)''$ ,  $L$  is amenable relative to  $B$  in  $N$ . By Theorem 5.2, we get either (i)'  $N \preceq_N (B \overline{\otimes} M_i)$  for some  $i$  or (ii)'  $N$  is amenable relative to  $B$  inside  $N$ . If (i)', by Lemma 2.2, one has  $M \preceq_M M_i$  which contradicts diffuseness of  $M_j$  (where  $i \neq j$ ) by Lemma 2.4. If (ii)', then we get that  $M$  is amenable by Lemma 2.8, which is a contradiction. Thus the condition (ii) does not happen.

Assume next condition (i). We have two conditions  $L \preceq_N (B \overline{\otimes} M_i)$  and  $L \not\preceq_N B$ , and it is known that they imply  $N = K \overline{\otimes} L \preceq_N (B \overline{\otimes} M_i)$  [IPP05]. Here we give a sketch of this argument in the paragraphs below for reader's convenience. Once we obtain it, then

by Lemma 2.2, this means  $M \preceq_M M_i$  which contradicts diffuseness of  $M_j$  (where  $i \neq j$ ) by Lemma 2.4, and hence we can end the proof.

Suppose now that  $L \preceq_N (B \overline{\otimes} M_1)$ . Then there is a  $*$ -homomorphism  $\theta: pLp \rightarrow q(B \overline{\otimes} M_1)q$  for some projections  $p \in L$ ,  $q \in B \overline{\otimes} M_1$ , and a partial isometry  $v \in N$  such that  $v\theta(x) = xv$  for  $x \in pLp$ . We may replace  $q$  with the support projection of  $E_{B \overline{\otimes} M_1}(v^*v)$ . Put  $D := \theta(pLp)$ . If  $D \preceq_{B \overline{\otimes} M_1} B$ , then by the choice of  $q$ , we can deduce  $L \preceq_{B \overline{\otimes} M} B$  (e.g. [Va08, Remark 3.8]) and hence a contradiction. So we have  $D \not\preceq_{B \overline{\otimes} M_1} B$ .

By [IPP05, Theorem 1.1], any quasi-normalizer of  $D$  in  $q(B \overline{\otimes} M)q$  is contained in  $B \overline{\otimes} M_1$ . In particular we have  $v^*v \in B \overline{\otimes} M_1$ . We put  $\tilde{q} := v^*v$ ,  $\tilde{\theta} := \theta(\cdot)\tilde{q}$ , and  $\tilde{D} := D\tilde{q}$ , and observe that  $\tilde{D} \not\preceq_{B \overline{\otimes} M_1} B$ . Write  $vv^* = pp'$  for some  $p' \in L' \cap N = K$ . Then we get a  $*$ -homomorphism  $\text{Ad } v^*: pLpp' \rightarrow \tilde{q}(B \overline{\otimes} M_1)\tilde{q}$ . Since  $v^*pLpp'v = \tilde{D} \not\preceq_{B \overline{\otimes} M_1} B$ , again by [IPP05, Theorem 1.1], any quasi-normalizer of  $v^*pLpp'v$  is contained in  $B \overline{\otimes} M_1$ . Hence we have  $v^*pp'Kpp'v \subset \tilde{q}(B \overline{\otimes} M_1)\tilde{q}$ . Thus we obtain  $v^*pp'(K \overline{\otimes} L)pp'v \subset \tilde{q}(B \overline{\otimes} M_1)\tilde{q}$  and  $K \overline{\otimes} L \preceq_N B \overline{\otimes} M_1$ . This is the desired condition.

## Case 2. $M$ is a wreath product group factor $L(\Delta \wr \Lambda)$ .

Write  $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$  for non-amenable, weakly amenable, and bi-exact groups  $\Lambda_i$ . For simplicity, we also write as  $\Gamma := \Delta \wr \Lambda = \Delta_\Lambda \rtimes \Lambda$ ,  $\hat{\Lambda}_i := \ker(\Lambda \rightarrow \Lambda_i)$ , and  $\hat{\Gamma}_i := \Delta_\Lambda \rtimes \hat{\Lambda}_i$  for all  $i$ .

Since  $K$  and  $L$  are regular subfactors, and  $K = L' \cap N$  and  $L = K' \cap N$ , by Theorem 5.3, it holds that  $K$  and  $L$  are amenable relative to  $B$  in  $M$ . They imply that  $K$  and  $L$  are amenable relative to  $B \overline{\otimes} L\hat{\Gamma}_i$  for all  $i$  (since  $B \subset B \overline{\otimes} L\hat{\Gamma}_i$ ). Regarding  $B \overline{\otimes} L\hat{\Gamma}_i$  as a crossed product of  $B \overline{\otimes} L\hat{\Gamma}_i$  by  $\Lambda_i$ , by Theorem 5.1, we get  $K \preceq_N B \overline{\otimes} L\hat{\Gamma}_i$  and  $L \preceq_N B \overline{\otimes} L\hat{\Gamma}_i$  for all  $i$ . Since  $K$  and  $L$  are regular in the factor  $N$ , they are equivalent to  $K \subset_{\text{approx}} B \overline{\otimes} L\hat{\Gamma}_i$  and  $L \subset_{\text{approx}} B \overline{\otimes} L\hat{\Gamma}_i$  for all  $i$  in the sense of [Va10, Definition 2.2] (see also [Va10, Proposition 2.6]). Hence by [Va10, Lemma 2.7], we have  $K \preceq_N B \overline{\otimes} L\Delta_\Lambda$  and  $L \preceq_N B \overline{\otimes} L\Delta_\Lambda$ . Finally by Lemma 2.5, we can deduce  $N = K \overline{\otimes} L \preceq_N B \overline{\otimes} L\Delta_\Lambda$ . By Lemma 2.3, this contradicts the fact that  $\Lambda$  is an infinite group.  $\square$

*Proof of Theorem A.* We consider only the case that  $M$  is the wreath product group factor, and other cases are proved by Theorem C and Proposition 3.6.

Put  $\Gamma := \Delta \wr \Lambda$  and  $M := L\Gamma$ . We will verify the sufficient condition in Lemma 3.4. Let  $B$  be a  $\text{II}_1$  factor and  $t > 0$  such that  $M \overline{\otimes} B \simeq M \overline{\otimes} B^t (= K \overline{\otimes} L)$ . Regarding  $M \overline{\otimes} B = K \overline{\otimes} L$ , we will show that either  $K \preceq_{M \overline{\otimes} B} B$ ,  $L \preceq_{M \overline{\otimes} B} B$ ,  $M \preceq_{M \overline{\otimes} B} L$ , or  $B \preceq_{M \overline{\otimes} B} L$ . So suppose by contradiction that any of them does not hold and we will deduce amenability of  $M$ , which is a contradiction.

We apply Theorem 5.3 to  $K$  and get either (i)  $K$  is amenable relative to  $B$  in  $M \overline{\otimes} B$  or (ii)  $K' \cap (M \overline{\otimes} B) = L \preceq_{M \overline{\otimes} B} B$ . So by assumption, we have that  $K$  is amenable relative to  $B$  in  $M \overline{\otimes} B$ . By the same reason,  $L$  is also amenable relative to  $B$  in  $M \overline{\otimes} B$ . Exchanging the roles of  $M \overline{\otimes} B$  and  $M \overline{\otimes} B^t$ , it further holds that  $M$  and  $B$  are amenable relative to  $L$  in  $M \overline{\otimes} B$ . Hence using  $M \prec_{M \overline{\otimes} B} L$  and  $L \prec_{M \overline{\otimes} B} B$  together with Proposition 2.7, we obtain that  $M$  is amenable relative to  $B$  in  $M \overline{\otimes} B$ . This means that  $M$  is amenable by Lemma 2.8 and thus a contradiction.  $\square$

## 6 Proof of Corollary B

Let  $G_n \in \mathcal{S}_{\text{factor}}$  for  $n \in \mathbb{N}$  (possibly  $G_n = G_m$  for different  $n, m$ ), and take  $\text{II}_1$  factors  $B_n$  with separable predual such that  $\mathcal{F}(B_n) = G_n$ . We may assume  $B_n = B_m$  whenever  $G_n = G_m$ . Let  $N$  be a free product  $\text{II}_1$  factor given by  $N := L\mathbb{F}_2 * L(\mathbb{Z} \rtimes \text{SL}(2, \mathbb{Z}))$ . Observe that  $\mathcal{F}(N) = \{1\}$  by [IPP05, Corollary 6.4] and hence  $\mathcal{F}(N \overline{\otimes} B_n) = \mathcal{F}(B_n)$  by

Theorem A. Define an infinite free product  $\text{II}_1$  factor  $M := \ast_{n=1}^{\infty} M_n$ , where  $M_n := N \overline{\otimes} B_n$  for all  $n \in \mathbb{N}$ . We first show that it satisfies  $\mathcal{F}(M) = \bigcap_{n \in \mathbb{N}} \mathcal{F}(B_n) = \bigcap_{n \in \mathbb{N}} G_n$ .

Recall first from [DR99, Theorem 1.5] that for any  $0 < t \leq 1$ , one has

$$M^t = \ast_{n=1}^{\infty} M_n^t$$

and this implies  $\bigcap_{n \in \mathbb{N}} \mathcal{F}(M_n) \subset \mathcal{F}(M)$  [DR99, Corollary 1.6]. Since  $\mathcal{F}(B_n) = \mathcal{F}(M_n)$  for all  $n \in \mathbb{N}$ , we get an inclusion  $\bigcap_{n \in \mathbb{N}} \mathcal{F}(B_n) \subset \mathcal{F}(M)$ .

We next see the reverse inclusion. Fix  $t \in \mathcal{F}(M)$ . Up to replacing with  $1/t$  if necessary, we may assume  $0 < t \leq 1$  and so we have an isomorphism

$$\ast_{n=1}^{\infty} M_n = M \simeq M^t = \ast_{n=1}^{\infty} M_n^t.$$

Since each  $M_n$  is a tensor product of non-amenable  $\text{II}_1$  factors, we can apply [HU15, Main Theorem] (see also [Oz04, IPP05, Po06b] for the case of finitely many free components). So there is a bijection  $\alpha$  on  $\mathbb{N}$  such that  $M_n$  and  $M_{\alpha(n)}^t$  are isomorphic for all  $n \in \mathbb{N}$ . Indeed, [HU15, Main Theorem] actually shows  $M_n \preceq_M M_{\alpha(n)}^t$  and  $M_{\alpha(n)}^t \preceq_M M_n$ . Once we get this condition, then by the proof of unique factorization of free products  $\text{II}_1$  factors (e.g. [Oz04, Theorem 3.3]) one can show that  $M_n$  and  $M_{\alpha(n)}^t$  are unitary conjugate in  $M$ , namely, there is  $u \in \mathcal{U}(M)$  such that  $uM_nu^* = M_{\alpha(n)}^t$  (under the given isomorphism). This particularly implies

$$G_n = \mathcal{F}(B_n) = \mathcal{F}(M_n) = \mathcal{F}(M_{\alpha(n)}^t) = \mathcal{F}(M_{\alpha(n)}) = \mathcal{F}(B_{\alpha(n)}) = G_{\alpha(n)}$$

and hence  $B_n = B_{\alpha(n)}$  by our choice of  $\{B_k\}_{k \in \mathbb{N}}$ . Thus the above isomorphism  $M_n \simeq M_{\alpha(n)}^t$  means  $t \in \mathcal{F}(M_n) = \mathcal{F}(B_n)$  for each  $n \in \mathbb{N}$ , and so  $t \in \bigcap_{n \in \mathbb{N}} \mathcal{F}(B_n)$ . We conclude  $\mathcal{F}(M) \subset \bigcap_{n \in \mathbb{N}} \mathcal{F}(B_n)$ .

Now we start the proof of Corollary B. The stability for intersection was already proved above. Let  $G \in \mathcal{S}_{\text{factor}}$  and take a  $\text{II}_1$  factor  $B$  with separable predual such that  $\mathcal{F}(B) = G$ . Then by putting  $B_n := B$  for all  $n \in \mathbb{N}$ , the above argument shows that  $\mathcal{F}(B) = \mathcal{F}(M)$  for  $M := \ast_{n \in \mathbb{N}} (B \overline{\otimes} N)$ , which is exactly the formula we mentioned in Introduction. Since  $M$  is a free product, it satisfies the property (TFF), so the first assertion of Corollary B holds. The stability for multiplication is then an immediate consequence of the first assertion and the definition of the property (TFF).

## 7 Some partial results

It would be interesting to know whether  $L(\mathbb{Z}^2 \rtimes \text{SL}(2, \mathbb{Z}))$  satisfies the property (TFF) or not. However we can not apply Theorem A because of the lacking of the weak amenability. In this section, we study some partial answers to this problem.

Observe that  $L(\mathbb{Z}^2 \rtimes \text{SL}(2, \mathbb{Z}))$  has two structures: one is the crossed product  $L^\infty(\mathbb{T}^2) \rtimes \text{SL}(2, \mathbb{Z})$  coming from a strongly ergodic action of a bi-exact weakly amenable group; and the other is a bi-exact group factor [Oz08]. From these viewpoints, we give partial answers to the property (TFF) as follows. See [BO08, Definition 12.3.9] for the definition of the  $W^*\text{CMAP}$  (or equivalently, the  $W^*\text{CBAP}$  with Cowling–Haagerup constant 1).

**Proposition 7.1.** *The following statements hold true.*

- (1) *Let  $\Gamma$  be a non-amenable, weakly amenable, and bi-exact group acting on a standard probability space  $X$  as a free, strongly ergodic, and p.m.p. action. Put  $M := L^\infty(X) \rtimes \Gamma$ . Then for any full  $\text{II}_1$  factor  $B$ , one has  $\mathcal{F}(B \overline{\otimes} M) = \mathcal{F}(B)\mathcal{F}(M)$ .*

- (2) Let  $\Gamma$  be a non-amenable bi-exact ICC group. Then for any  $\text{II}_1$  factor  $B$  with the  $W^*$ CMAP, one has  $\mathcal{F}(B \overline{\otimes} L\Gamma) = \mathcal{F}(B)\mathcal{F}(L\Gamma)$ .

The first assertion of this proposition will be proved by combining the proof of [Is14, Theorem 5.1.1] with the following lemma.

**Lemma 7.2** ([Ho15, Proposition 6.3]). *Let  $N = M \overline{\otimes} B = K \overline{\otimes} L$  be a tensor decomposition as  $\text{II}_1$  factors, and let  $A \subset M$  be a Cartan subalgebra. If  $K \preceq_N A \overline{\otimes} B$  and  $K$  is full, then we have  $K \preceq_N B$ .*

*Proof of Proposition 7.1(1).* We show that for any tensor decomposition  $M \overline{\otimes} B = K \overline{\otimes} L$  with  $B$  full, one has  $K \preceq_{M \overline{\otimes} B} B$  or  $L \preceq_{M \overline{\otimes} B} B$ . This gives the conclusion by Lemma 3.4.

Observe that  $M$  is full since the action is strongly ergodic. So by [Co75, Corollary 2.3], the tensor product  $M \overline{\otimes} B$  is full, and hence so are  $K$  and  $L$ . By Theorem 5.1 and the proof of [Is14, Theorem 5.1.1], one has  $K \preceq_{M \overline{\otimes} B} B \overline{\otimes} L^\infty(X)$  or  $L \preceq_{M \overline{\otimes} B} B \overline{\otimes} L^\infty(X)$ . Then we can apply Lemma 7.2, and obtain  $K \preceq_{M \overline{\otimes} B} B$  or  $L \preceq_{M \overline{\otimes} B} B$ .  $\square$

For the second assertion of Proposition 7.1, we prove the following proposition. This should be regarded as a “relativization” of Ozawa’s semisolidity theorem [Oz04, Theorem 4.6]. Actually we can not give a complete generalization of Ozawa’s theorem, since local reflexivity (or exactness) of  $C_\lambda^*(\Gamma)$  is not enough as an extension property in this setting. We will use the  $W^*$ CMAP on  $B$  to avoid this problem.

**Proposition 7.3.** *Let  $\Gamma$  be a bi-exact group and  $B$  a finite von Neumann algebra. Put  $M := L\Gamma \overline{\otimes} B$ . Then for any von Neumann subalgebra  $A \subset M$  with  $A \preceq_M B$ , there is a u.c.p. map from  $\langle M, B \rangle$  into  $A' \cap M$ , which restricts to the conditional expectation  $E_{A' \cap M}$  on  $L\Gamma \otimes_{\min} B$ .*

*If  $B$  has the  $W^*$ CMAP, then the resulting u.c.p. map can be taken as the one restricting  $E_{A' \cap M}$  on  $L\Gamma \overline{\otimes} B$ , and thus  $A' \cap M$  is amenable relative to  $B$ .*

**Proof.** Since most parts of the proof are straightforward “relativization” of the one of [Oz04, Theorem 4.6], we give only a sketch. Our proof here is very similar to the one of [BO08, Theorem 15.1.5] (and its generalization [Is12, Theorem 5.3.3]). We will use the Hilbert space  $H := L^2(M) \otimes_B L^2(M) = \ell^2(\Gamma) \otimes L^2(B) \otimes \ell^2(\Gamma)$ , in stead of  $L^2(M) \otimes L^2(M)$ .

The first part is exactly the same as the one of [BO08, Theorem 15.1.5] (and [Is12, Theorem 5.3.3]). Assume  $A \not\preceq_M B$ . By [BO08, Corollary F.14], we may assume  $A$  is abelian. Then one can define a proper conditional expectation

$$\Psi_A: \mathbb{B}(L^2(M)) \longrightarrow A' \cap \mathbb{B}(L^2(M)).$$

The condition  $A \not\preceq_M B$  implies  $\Psi_A(\mathbb{K}(\ell^2(\Gamma)) \otimes_{\min} \mathbb{B}(L^2(B))) = 0$ .

From now on, we use the relative tensor product. In particular, we will not use [Oz04, Proposition 4.2] but use a characterization of bi-exactness [BO08, Lemma 15.1.4]. Let  $\pi_H$  and  $\theta_H$  be left and right actions of  $M$  on  $H$ , and denote by  $\nu$  the algebraic  $*$ -homomorphism from  $\pi_H(M)\theta_H(M^{\text{op}})$  to  $\mathbb{B}(L^2(M))$  given by  $\nu(\pi_H(a)\theta_H(b^{\text{op}})) = ab^{\text{op}}$ . Let  $\Theta: C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma)^{\text{op}} \rightarrow \mathbb{B}(\ell^2(\Gamma))$  be a u.c.p. map such that  $\Theta(a \otimes b^{\text{op}}) - ab^{\text{op}} \in \mathbb{K}(\ell^2(\Gamma))$  [BO08, Lemma 15.1.4]. Put  $M_0 := C_\lambda^*(\Gamma) \otimes_{\min} B$ . Identifying  $C^*\{\pi_H(M_0), \theta_H(M_0)\}$  as  $C_\lambda^*(\Gamma) \otimes_{\min} C^*\{B, B^{\text{op}}\} \otimes_{\min} C_\lambda^*(\Gamma)^{\text{op}}$ , we may define  $\Theta$  on this algebra, which is the identity on  $B$  and  $B^{\text{op}}$ . Observe that at the  $C^*$ -algebra level,  $\Theta$  and  $\nu$  coincide modulo  $\mathbb{K}(\ell^2(\Gamma)) \otimes_{\min} \mathbb{B}(L^2(B))$ , that is,

$$\Theta(\pi_H(a)\theta_H(b^{\text{op}})) - ab^{\text{op}} \in \mathbb{K}(\ell^2(\Gamma)) \otimes_{\min} \mathbb{B}(L^2(B)), \quad a, b \in M_0.$$

Thus on  $C^*\{\pi_H(M_0), \theta_H(M_0)\}$  the composition  $\Phi_A \circ \nu$  coincides with  $\Phi_A \circ \Theta$ , and hence is a bounded u.c.p. map.

Observe that  $\Phi_A|_M$  is the unique trace preserving conditional expectation  $E_{A' \cap M}: M \rightarrow A' \cap M$ , and hence in particular *normal* on  $M$ . So the map  $\Phi_A \circ \nu$  is a normal u.c.p. map on  $\pi_H(M)$ . Regarding again  $C^*\{\pi_H(M_0), \theta_H(M_0)\} = C_\lambda^*(\Gamma) \otimes_{\min} C^*\{B, B^{\text{op}}\} \otimes_{\min} C_\lambda^*(\Gamma)^{\text{op}}$ , we can apply the local reflexivity of  $C_\lambda^*(\Gamma)$  (this comes from exactness of  $\Gamma$ ) and extend  $\Phi_A \circ \nu$  on  $L\Gamma \otimes_{\min} C^*\{B, B^{\text{op}}\} \otimes_{\min} C_\lambda^*(\Gamma)^{\text{op}}$  which is normal on  $L\Gamma$  (see Lemma 9.4.1, Proposition 9.2.5, and the proof of Lemma 9.2.9 in [BO08] for these facts). Finally by Arveson's extension theorem, we again extend  $\Phi_A \circ \nu$  on  $C^*\{\pi_H(\langle M, B \rangle), \theta_H(M_0)\}$ . Then the restriction on  $\pi_H(\langle M, B \rangle)$  of the resulting map defines a u.c.p. map from  $\langle M, B \rangle$  into  $A' \cap (M_0^{\text{op}})' = A' \cap M$ . By construction, this is a desired item.

Finally assume that  $B$  has the  $W^*$ CMAP, and take a net  $(\psi_i)_i$  of normal finite rank c.c. maps on  $B$  converging to  $\text{id}_B$  point weakly. We extend these maps to  $\langle M, B \rangle = \mathbb{B}(\ell^2(\Gamma)) \overline{\otimes} B$  by  $\text{id} \otimes \psi_i =: \tilde{\psi}_i$ . Observe that  $\tilde{\psi}_i(M) \subset L\Gamma \otimes_{\min} B$  for all  $i$ . Let  $\Phi$  be the u.c.p. map constructed in the first half of the proof. If we take a cluster point  $\tilde{\Phi}$  of  $(\Phi \circ \tilde{\psi}_i)_i$ , then this is a c.c. map from  $\langle M, B \rangle$  into  $A' \cap M$  which restricts to  $E_{A' \cap M}$  on  $M$ . In fact, for any  $x \in M \overline{\otimes} B$ , one has

$$\tilde{\Phi} \circ \tilde{\psi}_i(x) = E_{A' \cap M} \circ \tilde{\psi}_i(x) \rightarrow E_{A' \cap M}(x), \quad \text{as } i \rightarrow \infty.$$

Hence  $\tilde{\Phi}|_M = E_{A' \cap M}$  and  $\tilde{\Phi}$  is a conditional expectation onto  $A' \cap M$ .  $\square$

*Proof of Proposition 7.1(2).* Take  $t \in \mathcal{F}(L\Gamma \overline{\otimes} B)$  and fix  $M := L\Gamma \overline{\otimes} B = L\Gamma \overline{\otimes} B^t (= K \overline{\otimes} L)$ . By (the proof of) Lemma 3.4, we have only to show that  $K \preceq_M B$ ,  $L \preceq_M B$ ,  $L\Gamma \preceq_M L$ , or  $B \preceq_M L$ . So suppose by contradiction that each of them does not happen.

We apply Proposition 7.3 to  $K$  (actually an abelian subalgebra of  $K$  by [BO08, Corollary F.14]), and get that  $L \prec_M B$ . By exchanging the roles, we also have that  $L\Gamma \prec_M L$  and hence  $L\Gamma \prec_M B$  by Proposition 2.7. Thus by Lemma 2.8, we obtain amenability of  $L\Gamma$  which is a contradiction.  $\square$

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